λ-Adic Topology

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Would not it be great to live in a world in which every series $\Sigma_{n=1}^{\infty} a_n$ converges, as long as $a_n \to 0$? No failing those Calculus I tests, to start with! Well, this world is out there, and mathematicians and physicists alike have been using it at their convenience. This world is the world of λ-adic topology. In a simple case, this is a topology on the space $\mathbf{C}[\lambda]$ of polynomials in λ with complex coefficients. The λ-*adic topology* is defined by the following basis of neighborhoods of 0: $\mathbf{C}[\lambda] \supset \lambda\mathbf{C}[\lambda] \supset \lambda^2\mathbf{C}[\lambda] \supset \dots$ If you complete this topological space, *i.e.*, add the limits of all Cauchy sequences to it, you will get the space $\mathbf{C}[[\lambda]]$ of *formal power series in* λ. Every series $\Sigma_{n=0}^{\infty} a_n\lambda^n$ for any complex a_n will then converge in this space, actually because $a_n\lambda^n \to 0$ as $n \to \infty$ in the λ-adic topology. Given a prime number p, the ring of p-adic numbers $\mathbf{Q}_p$ may be obtained via a similar construction. A *p-adic number* is a formal power series $\Sigma_{n=k}^{\infty} a_n p^n$ with $a_n = 0, 1, \dots, p-1$ and k is some integer, which could well be negative. There is also an inverse-limit construction of the ring of power series and p-adic integers $\mathbf{Z}_p$, which is illustrated below.

Formal power series: $\mathbb{C}[[\lambda]] = \varprojlim_{n} \mathbb{C}[\lambda]/\lambda^n\mathbb{C}[\lambda]$

Formal Laurent series: $\mathbb{C}((\lambda)) = \varinjlim_{N} \lambda^{-N}\mathbb{C}[[\lambda]]$, $N>0$

For $f_0=f_0(\lambda), f_1=f_1(\lambda), \dots$ in $\mathbb{C}[[\lambda]]$, $\sum_{n=0}^{\infty} f_n$
converges in $\mathbb{C}[[\lambda]]$ as long as $f_n(\lambda) \to 0$ in $\mathbb{C}[[\lambda]]$

For example, $\sum_{n=0}^{\infty} a_n\lambda^n$ converges for any $a_0, a_1, \dots$ in $\mathbb{C}$

For a prime p: $\mathbb{Z}_p = \varprojlim_{n} \mathbb{Z}/p^n\mathbb{Z}$ p-adic integers

$\mathbb{Q}_p = \{ \frac{x}{p^N} \mid x \text{ in } \mathbb{Z}_p, N \geq 0 \}$ p-adic numbers